ISRAEL JOURNAL OF MATHEMATICS 91 (1995), 157–171

# BEHAVIOR OF DOMAIN CONSTANTS UNDER CONFORMAL MAPPINGS

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ABSTRACT

Domain constants are numbers attached to regions in the complex plane $\mathbb{C}$. For a region $\Omega$ in $\mathbb{C}$, let $d(\Omega)$ denote a generic domain constant. If there is an absolute constant $M$ such that $M^{-1} \le d(\Omega)/d(\Delta) \le M$ whenever $\Omega$ and $\Delta$ are conformally equivalent, then the domain constant is called quasi-invariant under conformal mappings. If $M = 1$, the domain constant is conformally invariant. There are several standard problems to consider for domain constants. One is to obtain relationships among different domain constants. Another is to determine whether a given domain constant is conformally invariant or quasi-invariant. In the latter case one would like to determine the best bound for quasi-invariance. We also consider a third type of result. For certain domain constants we show there is an absolute constant $N$ such that $|d(\Omega)-d(\Delta)| \le N$ whenever $\Omega$ and $\Delta$ are conformally equivalent, sometimes determining the best possible constant $N$. This distortion inequality is often stronger than quasi-invariance. We establish results of this type for six domain constants.

## 1. Introduction

Before discussing the six domain constants that we treat, we briefly recall some information about the hyperbolic and quasi-hyperbolic metrics. Notation for various domain constants is not standard; generally we employ the notation of [Ha] and [HM].

The hyperbolic metric on the unit disk $\mathbb{D} = \{z\colon |z| < 1\}$ is given by $\lambda_{\mathbb{D}}(z)|dz| = |dz|/(1-|z|^2)$. This metric is normalized to have Gaussian curvature $-4$. Let

* Research partially supported by a National Science Foundation Grant.
Received August 15, 1993

$\Omega$ be a hyperbolic region in the complex plane $\mathbb{C}$; that is, $\mathbb{C} \setminus \Omega$ contains at least two points. The hyperbolic metric $\lambda_\Omega(z)|dz|$ on $\Omega$ is determined from $\lambda_\Omega(\varphi(z))|\varphi'(z)| = \lambda_\mathbb{D}(z)$, where $\varphi \in \mathrm{Cov}(\mathbb{D}, \Omega)$, the class of holomorphic universal covering projections of $\mathbb{D}$ onto $\Omega$. This definition is independent of the choice of covering projection. Let $d_\Omega(a,b)$ be the hyperbolic distance between $a$ and $b$ on $\Omega$, and $D_\Omega(a,\rho) = \{z \in \Omega\colon d_\Omega(a,z) < \rho\}$ be the hyperbolic disk with center $a$ and hyperbolic radius $\rho$.

For a region $\Omega \neq \mathbb{C}$, the quasi-hyperbolic metric on $\Omega$ is $|dz|/\delta_\Omega(z)$, where $\delta_\Omega(z) = \mathrm{dist}(z, \partial\Omega)$ is the euclidean distance from $z$ to $\partial\Omega$. Schwarz's Lemma implies that $\lambda_\Omega(z)\delta_\Omega(z) \leq 1$ for every hyperbolic region $\Omega$ [K, p. 45].

(i) *The constant* $c(\Omega)$. A hyperbolic region $\Omega$ is called **uniformly perfect** if the hyperbolic and quasi-hyperbolic metrics are comparable [$P_2$]; that is, the domain constant

$$c(\Omega) = \inf\{\lambda_\Omega(z)\delta_\Omega(z)\colon z \in \Omega\}$$

is positive. It is known that $c(\Omega) \leq 1/2$ with equality if and only if $\Omega$ is convex (see [Hi], [HM]). Also, $1/4 \leq c(\Omega) \leq 1/2$ when $\Omega$ is simply connected.

The quasi-invariance of $c(\Omega)$ was established by Osgood [O]. He showed that there exists a positive constant $B \leq 6$ such that $1/B \leq c(\Delta)/c(\Omega) \leq B$ when $\Delta$ and $\Omega$ are conformally equivalent regions. Minda [$M_1$] decreased this to $B \leq 4\coth(\pi/2\sqrt{3}) = 5.5583\ldots$ . Harmelin and Minda [HM] improved this to $2 \leq B \leq (1+\coth^2(\pi/4))^{1/2} = 2.824\ldots$, and conjectured that $B = 2$. We obtain $B < (1+\coth^2(\pi/3))^{1/2} = 2.4335\ldots$ . We also establish that

$$\frac{1}{2(1+c(\Omega)-c(\Omega)^2)} \leq \frac{c(\Delta)}{c(\Omega)} \leq 2\left(1+c(\Delta)-c(\Delta)^2\right).$$

Note that as either $c(\Omega)$ or $c(\Delta)$ tends to zero, then the preceding upper and lower bounds both tend to 2, the conjectured constant. This shows that the conjectured value $B = 2$ is asymptotically true.

(ii) *The constant* $\eta(\Omega)$. For any hyperbolic region $\Omega$, the domain constant $\eta(\Omega)$ is defined by

$$\begin{aligned}\eta(\Omega) &= \frac{1}{2}\sup\left\{\lambda_\Omega^{-1}(w)|\nabla \log\lambda_\Omega(w)|\colon w \in \Omega\right\}\\ &= \sup\left\{\lambda_\Omega^{-1}(w)\left|\frac{\partial}{\partial w}\log\lambda_\Omega(w)\right|\colon w \in \Omega\right\}.\end{aligned}$$

We also know that

$$\eta(\Omega) = \sup\left\{\left|(1-|z|^2)\frac{\varphi''(z)}{2\varphi'(z)} - \bar{z}\right| : z \in \mathbb{D}\right\} \quad \text{for some } \varphi \in \mathrm{Cov}(\mathbb{D},\Omega)$$
$$= \sup\left\{\left|\frac{\varphi''(0)}{2\varphi'(0)}\right| : \varphi \in \mathrm{Cov}(\mathbb{D},\Omega)\right\}.$$

It is known that $\eta(\Omega) \geq 1$ and equality holds if and only if $\Omega$ is convex ([P$_1$], [Y$_1$]). (In [Y$_1$] the notation $\omega(\Omega)$ is used in place of $\eta(\Omega)$.) Moreover, $1 \leq \eta(\Omega) \leq 2$ when $\Omega$ is simply connected.

It was proved in [HM] that $1/2 \leq \eta(\Delta)/\eta(\Omega) \leq 2$ if $\Delta$ and $\Omega$ are conformally equivalent; the constants are best possible. This result implies that $\eta(\Delta)$ lies in an interval of length $3\eta(\Omega)/2$. As $\eta(\Omega)$ tends to $\infty$, the length of this interval also tends to $\infty$. We prove that for conformally equivalent regions $\Delta$ and $\Omega$, $|\eta(\Delta) - \eta(\Omega)| \leq 1$. This shows that $\eta(\Delta)$ lies in a closed interval of length 1 and yields the known result $1/2 \leq \eta(\Delta)/\eta(\Omega) \leq 2$.

We set

$$\tilde{\eta}(\Omega) = \sup\{\eta(\Delta) \colon \Delta \text{ is conformally equivalent to } \Omega\}.$$

Note that $\tilde{\eta}(\Omega) \leq \eta(\Omega) + 1 \leq 2\eta(\Omega)$ and $\tilde{\eta}(\Omega) = 2$ if $\Omega$ is simply connected because $\eta(\mathbb{C} \setminus (-\infty, 0]) = 2$.

(iii) *The constant* $a(\Omega)$. Let $S(\Omega)$ denote the family of functions holomorphic and univalent in $\Omega$ . Yamashita [Y$_1$] called a hyperbolic region $\Omega$ **of finite type** if the domain constant

$$a(\Omega) = \sup\left\{\frac{1}{\lambda_\Omega(z)}\left|\frac{f''(z)}{f'(z)}\right| : z \in \Omega, f \in S(\Omega)\right\}$$

is finite. Osgood [O] proved that $2/c(\Omega) \leq a(\Omega) \leq 4/c(\Omega)$, so $\Omega$ is of finite type if and only if it is uniformly perfect. In particular, $a(\Omega) \geq 4$; Osgood also showed that $a(\Omega) \leq 8$ if $\Omega$ is simply connected. In addition, Yamashita [Y$_1$] proved $a(\Omega) \leq 6$ for each convex region $\Omega$, and $a(\Omega) \leq 8\eta(\Omega)$.

We show that $a(\Omega) \leq 2\tilde{\eta}(\Omega) + 2\eta(\Omega)$ with equality if $\Omega$ is simply connected. This result contains those of Osgood and Yamashita mentioned above. We also prove that $a(\Omega)$ is quasi-invariant under conformal mappings: $1/2 \leq a(\Delta)/a(\Omega) \leq 2$ if $\Delta$ and $\Omega$ are conformally equivalent.

(iv) *The constant* $\beta(\Omega)$. If $\varphi \in \mathrm{Cov}(\mathbb{D},\Omega)$, then $\beta(\Omega)$ is defined by

$$\beta(\Omega) = \frac{1}{2}\sup\{(1-|z|^2)^2|S_\varphi(z)| \colon z \in \mathbb{D}\},$$

where

$$S_\varphi(z) = \frac{\varphi'''(z)}{\varphi'(z)} - \frac{3}{2}\left(\frac{\varphi''(z)}{\varphi'(z)}\right)^2$$

is the Schwarzian derivative of $\varphi$. If $\Delta$ and $\Omega$ are conformally equivalent, then $1/3 \le \beta(\Delta)/\beta(\Omega) \le 3$ provided $\beta(\Omega), \beta(\Delta) \ge 1$ and the constants are best possible [HM]. We show that if $\Delta$ and $\Omega$ are conformally equivalent, then $|\beta(\Delta) - \beta(\Omega)| \le 6$.

(v) *The constants $R(\Omega)$ and $R_c(\Omega)$.* For a holomorphic function $\varphi$ in $\mathbb{D}$, let

$$\rho(\varphi) = \sup\{\rho\colon \varphi \text{ is injective in } D_{\mathbb{D}}(a,\rho) \text{ for every } a \in \mathbb{D}\}$$

and

$$\rho_c(\varphi) = \sup\{\rho\colon \varphi \text{ is injective on } D_{\mathbb{D}}(a,\rho) \text{ and } \varphi(D_{\mathbb{D}}(a,\rho)) \text{ is convex in } \mathbb{C} \text{ for every } a \in \mathbb{D}\}.$$

The quantity $\rho(\varphi)$ is called the **hyperbolic radius of uniform local univalence** of $\varphi$, while $\rho_c(\varphi)$ is called the **hyperbolic radius of uniform local euclidean convexity** of $\varphi$. For $\varphi \in \mathrm{Cov}(\mathbb{D}, \Omega)$ we define $R(\Omega) = \tanh(\rho(\varphi))$ and $R_c(\Omega) = \tanh(\rho_c(\varphi))$. These two quantities are independent of the choice of the covering projection and so define domain constants; $R(\Omega)$ is a conformal invariant. $R(\Omega)$ is called the **pseudo-hyperbolic radius of injectivity** and $R_c(\Omega)$ is called the **pseudo-hyperbolic radius of convexity**. We know that [MM1]

$$\eta(\Omega) = \coth(2\rho_c(\varphi)) = \frac{1}{2}\left(R_c(\Omega) + \frac{1}{R_c(\Omega)}\right).$$

If $\Delta$ and $\Omega$ are conformally equivalent, then $2 - \sqrt{3} \le R_c(\Delta)/R_c(\Omega) \le 2 + \sqrt{3}$ and the constants are best possible [HM]. We shall also prove that if $\Delta$ and $\Omega$ are conformally equivalent, then

$$|R_c(\Delta) - R_c(\Omega)| \le \sqrt{3} - 1 \quad \text{and} \quad |1/R_c(\Delta) - 1/R_c(\Omega)| \le \sqrt{3} + 1;$$

both inequalities are best possible.

## 2. Additive change in $\beta(\Omega), \eta(\Omega)$ and $R_c(\Omega)$ under a conformal mapping

Harmelin and Minda [HM] obtained the best possible bounds on the ratios $\beta(\Delta)/\beta(\Omega), \eta(\Delta)/\eta(\Omega)$ and $R_c(\Delta)/R_c(\Omega)$ for conformally equivalent $\Delta$ and $\Omega$. In this section, we give constant upper bounds on the differences

$$|\beta(\Delta) - \beta(\Omega)|, \quad |\eta(\Delta) - \eta(\Omega)|, \quad |R_c(\Delta) - R_c(\Omega)| \quad \text{and} \quad \left|\frac{1}{R_c(\Delta)} - \frac{1}{R_c(\Omega)}\right|;$$

except possibly for the first difference, the constant bound is best possible.

THEOREM 1: *Let $\Delta$ and $\Omega$ be conformally equivalent. Then $|\beta(\Delta) - \beta(\Omega)| \le 6$.*

*Proof:* Since $\Delta$ and $\Omega$ are conformally equivalent, there exists $f \in S(\Omega)$ such that $\Delta = f(\Omega)$. Let $\varphi \in \mathrm{Cov}(\mathbb{D}, \Omega)$. Then $g = f \circ \varphi \in \mathrm{Cov}(\mathbb{D}, \Delta)$ and $S_g(z) = S_f(\varphi(z))\varphi'(z)^2 + S_\varphi(z)$. Note that $\lambda_\Omega(\varphi(z))|\varphi'(z)| = 1/(1-|z|^2)$. We have

$$\begin{aligned}
\beta(\Delta) &= \frac{1}{2}\sup\left\{\left(1-|z|^2\right)^2|S_g(z)|: z \in \mathbb{D}\right\} \\
&\le \frac{1}{2}\sup\left\{\left(1-|z|^2\right)^2|\varphi'(z)|^2|S_f(\varphi(z))|: z \in \mathbb{D}\right\} \\
&\quad + \frac{1}{2}\sup\left\{\left(1-|z|^2\right)^2|S_\varphi(z)|: z \in \mathbb{D}\right\} \\
&= \frac{1}{2}\sup\left\{\lambda_\Omega^{-2}(w)|S_f(w)|: w \in \Omega\right\} + \beta(\Omega) \\
&\le 6 + \beta(\Omega).
\end{aligned}$$

The last inequality follows from a result of Beardon and Gehring [BG]. Thus, $\beta(\Delta) - \beta(\Omega) \le 6$. By symmetry, $\beta(\Omega) - \beta(\Delta) \le 6$.

If $\Delta$ and $\Omega$ are simply connected, then $0 \le \beta(\Omega), \beta(\Delta) \le 3$ and the upper bound in Theorem 1 can be replaced by 3.

Yamashita [$\mathrm{Y}_1$] was the first to consider bounding the difference of $\eta(\Delta)$ and $\eta(\Omega)$ for conformally equivalent regions; he proved that $|\eta(\Delta) - \eta(\Omega)| \le a(\Omega)/2$. The inequalities $\beta(\Delta) \ge \eta(\Delta)^2 - 1$ [$\mathrm{P}_1$] and $\beta(\Omega) \le \eta(\Omega)^2 + 1$ [Ha] in conjunction with Theorem 1 yield $|\eta(\Delta) - \eta(\Omega)| \le 8/(\eta(\Delta) + \eta(\Omega))$ for conformally equivalent $\Omega$ and $\Delta$, which tells us that the quantity $|\eta(\Delta) - \eta(\Omega)|$ tends to zero as either $\eta(\Delta)$ or $\eta(\Omega)$ tends to $\infty$. We now obtain the sharp constant upper bound on this difference.

THEOREM 2: *Suppose $\Delta$ and $\Omega$ are conformally equivalent. Then*

$$|\eta(\Delta) - \eta(\Omega)| \le 1.$$

*This bound is best possible.*

*Proof:* We begin by demonstrating that the bound is best possible. The sharpness is clear from $\eta(\mathbb{D}) = 1$ and $\eta(\mathbb{C} \setminus (-\infty, 0]) = 2$.

It is enough to show that $\eta(\Delta) - \eta(\Omega) \leq 1$ for conformally equivalent regions $\Omega$ and $\Delta$. If $\Omega$ and $\Delta$ are simply connected, then $\eta(\Delta)$, $\eta(\Omega) \in [1, 2]$ and the inequality is trivial in this case. Therefore, we may assume that $\Omega$ and $\Delta$ are not simply connected, so that $\eta(\Omega) > 1$ and $\beta(\Omega) > 1$. We consider several cases.

First, assume $\eta(\Omega) \geq 3.5$. Suppose it were true that $\eta(\Delta) > 1 + \eta(\Omega)$. Now, $\beta(\Delta) \geq \eta(\Delta)^2 - 1$ [P$_1$] and $\beta(\Omega) \leq \eta(\Omega)^2 + 1$ [Ha]. By using Theorem 1 we have

$$1 < \eta(\Delta) - \eta(\Omega) = \frac{\eta^2(\Delta) - \eta^2(\Omega)}{\eta(\Delta) + \eta(\Omega)} < \frac{\beta(\Delta) + 1 - \beta(\Omega) + 1}{1 + 2\eta(\Omega)} \leq \frac{6 + 2}{1 + 7} = 1,$$

a contradiction. Therefore, $\eta(\Delta) - \eta(\Omega) \leq 1$ when $\eta(\Omega) \geq 3.5$.

It remains to consider the case in which $1 < \eta(\Omega) < 3.5$ and $\beta(\Omega) > 1$. We know that

$$\eta(\Delta) \leq \sqrt{1 + 3/R^2(\Delta)} \text{ [MM}_1\text{]} \quad \text{and} \quad R(\Omega) \geq \tanh\left(\frac{\pi}{2\sqrt{\beta(\Omega) - 1}}\right)$$

(see [BS], [M$_1$], [S] ), so

$$\begin{aligned} (1) \qquad \eta(\Delta) - \eta(\Omega) &\leq \sqrt{1 + 3/R^2(\Delta)} - \eta(\Omega) = \sqrt{1 + 3/R^2(\Omega)} - \eta(\Omega) \\ &\leq \sqrt{1 + 3\coth^2\left(\frac{\pi}{2\sqrt{\beta(\Omega) - 1}}\right)} - \eta(\Omega). \end{aligned}$$

Suppose $\sqrt{1 + \sqrt{2}} \leq \eta(\Omega) < 3.5$. By using $\beta(\Omega) \leq \eta(\Omega)^2 + 1$ [Ha], we have

$$\eta(\Delta) - \eta(\Omega) \leq \sqrt{1 + 3\coth^2\left(\frac{\pi}{2\eta(\Omega}\right)} - \eta(\Omega).$$

We prove that the right-hand side of the preceding inequality is at most 1. This is equivalent to

$$F(\eta) \equiv \sqrt{\frac{\eta^2 + 2\eta}{3}} - \coth\left(\frac{\pi}{2\eta}\right) \geq 0, \quad \sqrt{1 + \sqrt{2}} \leq \eta < 3.5.$$

Note that

$$F'(\eta) = \frac{\eta + 1}{\sqrt{3(\eta^2 + 2\eta)}} - \frac{\pi}{2\eta^2 \sinh^2(\pi/2\eta)}.$$

It is easy to see that both $(\eta+1)/\sqrt{3(\eta^2+2\eta)}$ and $\eta\sinh(\pi/2\eta)$ are decreasing on $(1,\infty)$, so $F(\eta)$ is concave downward on $(1,\infty)$. As $F(\sqrt{1+\sqrt{2}})>0$ and $F(3.5)=\sqrt{77/12}-\coth(\pi/7)>0$, we conclude that $F(\eta)>0$ on $[\sqrt{1+\sqrt{2}},3.5)$.

Finally, we consider the most delicate case: $1<\eta(\Omega)<\sqrt{1+\sqrt{2}}$. From (1), it is enough to show that

$$\sqrt{1+3\coth^2\left(\frac{\pi}{2\sqrt{\beta(\Omega)-1}}\right)}-\eta(\Omega)\le 1,$$

which is equivalent to

$$\frac{\pi}{\sqrt{\beta(\Omega)-1}}-\log\frac{\sqrt{\eta^2(\Omega)+2\eta(\Omega)}+\sqrt{3}}{\sqrt{\eta^2(\Omega)+2\eta(\Omega)}-\sqrt{3}}\ge 0.$$

For $\eta(\Omega)\in(1,\sqrt{1+\sqrt{2}}]=I$, Harmelin [Ha] proved that

$$\beta(\Omega)\le\frac{1}{2^{3/2}\eta}\left(27\eta^4-18\eta^2-1+\left(\eta^2-1\right)^{1/2}\left(9\eta^2-1\right)^{3/2}\right)^{1/2},$$

where $\eta=\eta(\Omega)$. The result will follow if we can show that

$$\frac{\pi 2^{3/4}\sqrt{\eta}}{(\sqrt{27\eta^4-18\eta^2-1+(\eta^2-1)^{1/2}(9\eta^2-1)^{3/2}}-2^{3/2}\eta)^{1/2}}$$
$$-\log\frac{\sqrt{\eta^2+2\eta}+\sqrt{3}}{\sqrt{\eta^2+2\eta}-\sqrt{3}}\ge 0\quad\text{for }\eta\in I.$$

Let $G(\eta)$ denote the left-hand side of the inequality. The minimum value of $G$ on $I$ is $0.063357\ldots$ and is attained uniquely at the point $x=1.182290\ldots$. This was verified using both *Mathematica 2.0* and *Maple V*. This completes the proof.

*Remark:* Yamashita [$\mathrm{Y}_1$] observed that

$$N(\Omega)=\{\eta(\Delta)\colon \Delta\ \text{is conformally equivalent to}\ \Omega\}$$

is the closed interval $[1,2]$ when $\Omega$ is simply connected. He raised the problem of determining the set $N(\Omega)$ for uniformly perfect, but not simply connected, $\Omega$. He pointed out that the set is contained in the interval $(1,\infty)$. Theorem 2 shows that $N(\Omega)$ is always contained in a closed interval of length 1; actually, the length of the interval tends to zero as $\eta(\Omega)$ tends to $\infty$.

THEOREM 3: *Let $\Delta$ and $\Omega$ be conformally equivalent. Then*

$$|R_c(\Delta) - R_c(\Omega)| \le \sqrt{3} - 1$$

*and*

$$\left|\frac{1}{R_c(\Delta)} - \frac{1}{R_c(\Omega)}\right| \le \sqrt{3} + 1.$$

*These upper bounds are best possible.*

*Proof:* In order to establish the first inequality, we need only prove that $R_c(\Delta) - R_c(\Omega) \le \sqrt{3} - 1$. Since $R_c(\Delta) \le R(\Delta)$ and $R_c(\Omega) \ge (2 - \sqrt{3})R(\Omega)$ [HM], we obtain

$$R_c(\Delta) - R_c(\Omega) \le R(\Delta) - (2 - \sqrt{3})R(\Omega) = (\sqrt{3} - 1)R(\Omega) \le \sqrt{3} - 1.$$

For $\Delta = \mathbb{D}$ and $\Omega = \mathbb{C} \setminus (-\infty, 0], R_c(\Delta) = 1$ and $R_c(\Omega) = 2 - \sqrt{3}$. Thus, the estimate is sharp.

The second inequality is a consequence of the first and Theorem 2. Because

$$\eta(\Omega) = \frac{1}{2}\left(R_c(\Omega) + \frac{1}{R_c(\Omega)}\right),$$

we have

$$\left|\frac{1}{R_c(\Delta)} - \frac{1}{R_c(\Omega)}\right| \le 2|\eta(\Delta) - \eta(\Omega)| + |R_c(\Delta) - R_c(\Omega)| \le 2 + \sqrt{3} - 1 = \sqrt{3} + 1.$$

The same $\Delta$ and $\Omega$ as before show that this inequality is best possible.

## 3. Quasi-invariance of $c(\Omega)$

In order to obtain better upper and lower bounds on $c(\Delta)/c(\Omega)$ for conformally equivalent $\Delta$ and $\Omega$, we improve the estimate $\eta(\Omega) \le 1/c(\Omega)$ of Osgood [O]. Recall that a subset $E$ of a hyperbolic region $\Omega$ is **hyperbolically convex** in $\Omega$ if for every pair of points $a$ and $b$ in $E$ each arc of any hyperbolic geodesic in $\Omega$ connecting $a$ and $b$ lies in $E$ (see [J] and [$\mathrm{MM}_2$]). Osgood [O] showed that $|\nabla \log \lambda_\Omega(w)| \le 2/\delta_\Omega(w)$ for any hyperbolic region; he raised the question of the sharpness of the constant 2. For simply connected regions the sharp constant is 4/3 [O], while it can be reduced to 1 for convex regions [$\mathrm{M}_2$]. The following result strengthens Osgood's inequality and shows that the constant 2 can be quantitatively improved for uniformly perfect regions. Our method of proof is different from Osgood's.

THEOREM 4: *For any hyperbolic region* $\Omega$,

$$|\nabla \log \lambda_\Omega(w)| \le \frac{2}{\delta_\Omega(w)}\left[1-\lambda_\Omega^2(w)\delta_\Omega^2(w)\right] \le \frac{2}{\delta_\Omega(w)}\left[1-c^2(\Omega)\right].$$

*Proof:* The second inequality is elementary. The first inequality is established as follows. Fix $a \in \Omega$. Let $\varphi \in \mathrm{Cov}(\mathbb{D},\Omega), a=\varphi(0)$ and $\delta=\delta_\Omega(a)$. Then $\lambda_\Omega(a)|\varphi'(0)| = \lambda_{\mathbb{D}}(0) = 1$. Choose $f(w)=\varphi^{-1}(a+\delta w)$ with $f(0)=0$; a single-valued branch of $\varphi^{-1}$ exists in the euclidean disk $D(a,\delta)$ with center $a$ and radius $\delta$ since $\varphi$ is a covering projection. We have $\varphi'(0)f'(0)=\delta$ and $\varphi''(0)f'(0)^2+\varphi'(0)f''(0)=0$. Because $D(a,\delta)$ is hyperbolically convex in $\Omega$ [J], the function $f$ is hyperbolically convex in $\mathbb{D}$. Since $f$ is hyperbolically convex in $\mathbb{D}$, $|f''(0)| \le 2|f'(0)|(1-|f'(0)|^2)$ [MM$_2$]. This yields

$$\frac{|\nabla \log \lambda_\Omega(a)|}{2\lambda_\Omega(a)} = \left|\frac{\varphi''(0)}{2\varphi'(0)}\right| = \left|\frac{f''(0)}{2f'(0)^2}\right| \le \frac{1}{|f'(0)|} - |f'(0)|$$

$$= \frac{|\phi'(0)|}{\delta} - \frac{\delta}{|\phi'(0)|} = \frac{1}{\lambda_\Omega(a)\delta_\Omega(a)} - \lambda_\Omega(a)\delta_\Omega(a),$$

which is the desired result.

COROLLARY 1: *For any hyperbolic region* $\Omega, \eta(\Omega) \le \dfrac{1}{c(\Omega)} - c(\Omega)$.

*Proof:* Since the function $h(t)=t-1/t$ is decreasing on $(0,1)$ and $c(\Omega) \le \lambda_\Omega(a)\delta_\Omega(a)$, the result follows from the inequality in the proof of Theorem 4.

COROLLARY 2: *If* $\Delta$ *and* $\Omega$ *are conformally equivalent hyperbolic regions, then*

$$\frac{1}{2\left(1+c(\Omega)-c(\Omega)^2\right)} \le \frac{c(\Delta)}{c(\Omega)} \le 2\left(1+c(\Delta)-c(\Delta)^2\right).$$

*Proof:* We prove the upper bound; the lower bound then follows by symmetry. By using Corollary 1 for $\Delta$, Theorem 2 and $1/c(\Omega) \le 2\eta(\Omega)$ [O], we see that

$$\frac{1}{c(\Omega)} \le 2\eta(\Omega) \le 2(\eta(\Delta)+1) \le 2\left(\frac{1}{c(\Delta)} - c(\Delta) + 1\right),$$

which is equivalent to

$$\frac{c(\Delta)}{c(\Omega)} \le 2\left(1+c(\Delta)-c(\Delta)^2\right).$$

*Remark:* There is an interesting geometric interpretation for Theorem 4. Jørgensen [J] proved that if $\gamma$ is a hyperbolic geodesic in $\Omega$ and $C_w$ is the euclidean circle of curvature at a point $w \in \gamma$, then $\overline{C_w} \cap \partial\Omega \ne \emptyset$. Theorem 4 strengthens this to show that the euclidean circle of curvature actually protrudes beyond $\partial\Omega$ when $\Omega$ is uniformly perfect. The hyperbolic curvature $\kappa_\Omega(w,\gamma)$ of a path $\gamma$: $w = w(t)$ in $\Omega$ is given by

$$\kappa_\Omega(w,\gamma)\lambda_\Omega(w) = \kappa_e(w,\gamma) + \mathrm{Im}\left\{\frac{2\partial \log \lambda_\Omega(w)}{\partial w}\frac{w'(t)}{|w'(t)|}\right\},$$

where $\kappa_e(w,\gamma)$ is the euclidean curvature. A hyperbolic geodesic has zero hyperbolic curvature, so the euclidean curvature satisfies

$$|\kappa_e(w,\gamma)| \le \left|\mathrm{Im}\left\{\frac{2\partial \log \lambda_\Omega(w)}{\partial w}\frac{w'(t)}{|w'(t)|}\right\}\right| \le \frac{2}{\delta_\Omega(w)}[1-c^2(\Omega)].$$

If $r_e(w,\gamma)$ is the euclidean radius of curvature, the preceding inequality yields $r_e(w,\gamma) \ge \delta_\Omega(w)/2\left[1-c^2(\Omega)\right]$. Thus, the diameter of $C_w$ is at least $\delta_\Omega(w)/[1-c^2(\Omega)] > \delta_\Omega(w)$, so $C_w$ must contain boundary points in its interior.

Corollary 2 immediately yields $c(\Delta)/c(\Omega) \le 2.5$ for conformally equivalent regions $\Delta$ and $\Omega$. We can do even better.

THEOREM 5: *Suppose $\Delta$ and $\Omega$ are conformally equivalent. Then*

$$\frac{1}{\sqrt{1+3\coth^2(\pi/3)}} < \frac{c(\Delta)}{c(\Omega)} < \sqrt{1+3\coth^2(\pi/3)} = 2.4335\ldots.$$

*Proof:* We only need to establish the lower bound because of symmetry. If $\Omega$ and $\Delta$ are simply connected, then the inequality is trivial since $c(\Omega)$, $c(\Delta) \in [1/4,\ 1/2]$. Therefore, we may assume $\Omega$ and $\Delta$ are not simply connected. From $\eta(\Delta) \le \sqrt{1+3/R^2(\Delta)}$ [MM$_1$] and $c(\Delta) \ge 1/2\eta(\Delta)$ [O], we see that

$$c(\Delta) \ge \frac{R(\Delta)}{2\sqrt{3+R^2(\Delta)}} = \frac{R(\Omega)}{2\sqrt{3+R^2(\Omega)}}.$$

The inequality $R(\Omega) \ge \tanh(\pi/2\eta(\Omega))$ is valid if $\beta(\Omega) \ge 1$ [HM]; $\beta(\Omega) \ge 1$ holds because $\Omega$ is not simply connected [N]. Corollary 1 of Theorem 4 now gives

$$\frac{c(\Delta)}{c(\Omega)} \ge \frac{1}{2c(\Omega)\sqrt{1+3\coth^2\left(\dfrac{\pi c(\Omega)}{2-2c(\Omega)^2}\right)}}.$$

We obtain the desired lower bound if we can prove for $c(\Omega) \in [0, 1/2)$ that

$$2c(\Omega)\sqrt{1 + 3\coth^2\left(\frac{\pi c(\Omega)}{2 - 2c(\Omega)^2}\right)} < \sqrt{1 + 3\coth^2(\pi/3)};$$

the right-hand side of the inequality is the value of the left-hand side for $c(\Omega) = 1/2$. As $c(\Omega)$ tends to zero, it is easy to see that the left-hand side of the preceding inequality tends to $4\sqrt{3}/\pi$, which is strictly less than $\sqrt{1 + 3\coth^2(\pi/3)}$. So we only need to consider the case $c(\Omega) \in (0, 1/2)$. Direct computation shows that the above inequality is the same as

$$\tanh\left(\frac{\pi x}{2 - 2x^2}\right) > \frac{2\sqrt{3}x}{\sqrt{a - 4x^2}},$$

where $x = c(\Omega)$ and $a = 1 + 3\coth^2(\pi/3)$, or equivalently,

$$h(x) \equiv \frac{\pi x}{1 - x^2} - \log \frac{\sqrt{a - 4x^2} + 2\sqrt{3}x}{\sqrt{a - 4x^2} - 2\sqrt{3}x} > 0.$$

Now,

$$h'(x) = \frac{\pi(1 + x^2)}{(1 - x^2)^2} - \frac{4\sqrt{3}a}{\sqrt{a - 4x^2}(a - 16x^2)}.$$

If we can prove that $h'(x)$ has only one zero $x_0$ in $(0, 1/2)$, then from

$$h'(0) = \pi - \frac{4\sqrt{3}}{\sqrt{a}} > 0 \quad \text{and} \quad h'(1/2) = \frac{20\pi}{9} - \frac{4\sqrt{3}a}{\sqrt{a - 1}(a - 4)} < 0$$

we know that $h'(x) > 0$ on $[0, x_0)$ and $h'(x) < 0$ on $(x_0, 1/2]$. Therefore, $h(x) > 0$ in $(0, 1/2)$ since $h(0) = h(1/2) = 0$. Set $y = 4x^2$, then $0 < y < 1$ and $h'(x) = 0$ is the same as

$$g(y) \equiv \pi^2(4 + y)^2(a - 4y)^2(a - y) - 3a^2(4 - y)^4 = 0.$$

As $g(0) = 16a^2(\pi^2 a - 48) > 0$, $g(1) = 25\pi^2(a - 4)^2(a - 1) - 243a^2 < 0$ and $g'(y) < 0$ on $(0, 1)$, which can be checked by writing $g'(y)$ as a polynomial, we conclude that $g(y)$ has only one zero on $(0, 1)$. Thus, $h'(x)$ has only one zero on $(0, 1/2)$. This completes the proof of Theorem 5.

## 4. Quasi-invariance of $a(\Omega)$

We investigate estimates of $a(\Delta)/a(\Omega)$ for conformally equivalent $\Delta$ and $\Omega$. First, we establish the following upper bound on $a(\Omega)$ in terms of $\eta(\Omega)$ and $\tilde{\eta}(\Omega)$.

THEOREM 6: *For a hyperbolic region $\Omega$, $a(\Omega) \le 2\tilde{\eta}(\Omega) + 2\eta(\Omega)$ with equality if $\Omega$ is simply connected.*

*Proof:* Note that $\eta(\Omega)$, $\tilde{\eta}(\Omega)$ and $a(\Omega)$ can be expressed, respectively, by

$$\eta(\Omega) = \sup\left\{\left|\frac{\varphi''(0)}{2\varphi'(0)}\right| : \varphi \in \mathrm{Cov}(\mathbb{D},\Omega)\right\},$$

$$\tilde{\eta}(\Omega) = \sup\left\{\left|\frac{(f\circ\varphi)''(0)}{2(f\circ\varphi)'(0)}\right| : \varphi \in \mathrm{Cov}(\mathbb{D},\Omega) \text{ and } f \in S(\Omega)\right\}$$

and

$$\begin{aligned} a(\Omega) &= \sup\left\{\lambda_\Omega^{-1}(\varphi(z))\left|\frac{f''(\varphi(z))}{f'(\varphi(z))}\right| : z \in \mathbb{D} \text{ and } f \in S(\Omega)\right\} \\ &\qquad\qquad \text{for some } \varphi \in \mathrm{Cov}(\mathbb{D},\Omega) \\ &= \sup\left\{(1-|z|^2)\left|\frac{(f\circ\varphi)''(z)}{(f\circ\varphi)'(z)} - \frac{\varphi''(z)}{\varphi'(z)}\right| : z \in \mathbb{D} \text{ and } f \in S(\Omega)\right\} \\ &\qquad\qquad \text{for some } \varphi \in \mathrm{Cov}(\mathbb{D},\Omega) \\ &= \sup\left\{\left|\frac{(f\circ\varphi)''(0)}{(f\circ\varphi)'(0)} - \frac{\varphi''(0)}{\varphi'(0)}\right| : \varphi \in \mathrm{Cov}(\mathbb{D},\Omega) \text{ and } f \in S(\Omega)\right\}. \end{aligned}$$

It is clear that $a(\Omega) \le 2\tilde{\eta}(\Omega) + 2\eta(\Omega)$.

Next we prove the opposite inequality for simply connected $\Omega$. For any fixed $\epsilon > 0$, there exists $\varphi \in \mathrm{Cov}(\mathbb{D},\Omega)$ such that $\varphi''(0)/\varphi'(0) < -2\eta(\Omega) + \epsilon$. This is true because for each $\epsilon > 0$ there is $\varphi \in \mathrm{Cov}(\mathbb{D},\Omega)$ such that $|\varphi''(0)/\varphi'(0)| > 2\eta(\Omega) - e$, and we may assume that $\varphi''(0)/\varphi'(0) < 0$ since otherwise we can replace $\varphi(z)$ by $\varphi(e^{i\theta}z)$ for some $\theta \in \mathbb{R}$, which also belongs to $\mathrm{Cov}(\mathbb{D},\Omega)$. If $\Omega$ is simply connected, then $\varphi$ is univalent. Set $f = k \circ \varphi^{-1} \in S(\Omega)$, where $k(z) = z(1-z)^{-2}$ is the Koebe function. As $(f\circ\varphi)''(0)/(f\circ\varphi)'(0) = k''(0)/k'(0) = 4, a(\Omega) > 4 + 2\eta(\Omega) - \epsilon$. Since $\tilde{\eta} = 2$ for simply connected $\Omega$ and $\epsilon$ is arbitrary, we must have $a(\Omega) \ge 2\tilde{\eta}(\Omega) + 2\eta(\Omega)$.

Notice that if $\Omega$ is simply connected, then Theorem 6 gives $a(\Omega) = 4 + 2\eta(\Omega) \in [6, 8]$. This yields Osgood's inequality $a(\Omega) \le 8$ for $\Omega$ simply connected. For $\Omega$

convex this gives $a(\Omega) = 6$ after appealing to Yamashita's result $a(\Omega) \leq 6$ when $\Omega$ is convex [$\text{Y}_1$]. Does the stronger result that $a(\Omega) \geq 6$ for any hyperbolic region with equality if and only if $\Omega$ is convex hold? We note that $a(\Omega) > 4$ in general. We know $a(\Omega) \geq 2/c(\Omega) \geq 4$. If $a(\Omega) = 4$, then $c(\Omega) = 1/2$, which would imply that $\Omega$ is convex ([Hi], [HM]). But for convex $\Omega$, $a(\Omega) = 6$. This is a contradiction, so $a(\Omega) > 4$. The next result implies that $a(\Omega) \geq 6$ when $\eta(\Omega) \geq 3$.

COROLLARY 1: *For any hyperbolic region* $\Omega$, $2\eta(\Omega) \leq a(\Omega) \leq 4\eta(\Omega) + 2$. *The upper bound is best possible.*

*Proof:* The lower bound follows from $a(\Omega) \geq 2/c(\Omega)$ and $1/c(\Omega) \geq \eta(\Omega)$ [O], while Theorem 2 implies that $\tilde{\eta}(\Omega) \leq \eta(\Omega) + 1$, which yields the upper bound from Theorem 6. Equality holds in the upper bound for any convex region.

Yamashita [$\text{Y}_1$, Prop. 3] showed that $a(\Omega) \leq 8\eta(\Omega)$; Corollary 1 improves this.

COROLLARY 2: *If* $\Omega$ *is a hyperbolic region, then* $a(\Omega) \leq 4\sqrt{1 + 3/R^2(\Omega)}$. *Equality holds for* $\Omega = \mathbb{C} \setminus (-\infty, 0]$.

*Proof:* Since $R(\Omega)$ is conformally invariant and $\eta(\Omega) \leq \sqrt{1 + 3/R^2(\Omega)}$ [$\text{MM}_1$], we have $\tilde{\eta}(\Omega) \leq \sqrt{1 + 3/R^2(\Omega)}$. Theorem 6 then yields $a(\Omega) \leq 4\sqrt{1 + 3/R^2(\Omega)})$.

The weaker inequality $a(\Omega) \leq 8/R(\Omega)$ follows from work of Yamashita [$\text{Y}_2$] who established a stronger pointwise result.

THEOREM 7: *Let* $\Delta$ *and* $\Omega$ *be conformally equivalent. Then* $1/2 \leq a(\Delta)/a(\Omega) \leq 2$.

*Proof:* It is enough to show that $a(\Delta) \leq 2a(\Omega)$. Let $g \in S(\Omega)$ be such that $\Delta = g(\Omega)$. Then $F \in S(\Delta)$ if and only if $f = F \circ g \in S(\Omega)$. Note that

$$\frac{F''(g(z))}{F'(g(z))} g'(z) = \frac{f''(z)}{f'(z)} - \frac{g''(z)}{g'(z)}.$$

We have

$$\begin{aligned}
a(\Delta) &= \sup\left\{\lambda_\Delta^{-1}(w)\left|\frac{F''(w)}{F'(w)}\right| : w\in\Delta \text{ and } F\in S(\Delta)\right\}\\
&= \sup\left\{\lambda_\Omega^{-1}(z)\left|\frac{f''(z)}{f'(z)}-\frac{g''(z)}{g'(z)}\right| : z\in\Omega \text{ and } f\in S(\Omega)\right\}\\
&\le \sup\left\{\lambda_\Omega^{-1}(z)\left|\frac{f''(z)}{f'(z)}\right| : z\in\Omega \text{ and } f\in S(\Omega)\right\}\\
&\quad + \sup\left\{\lambda_\Omega^{-1}(z)\left|\frac{g''(z)}{g'(z)}\right| : z\in\Omega\right\}\\
&\le 2a(\Omega).
\end{aligned}$$

What are the best constants in Theorem 7? If $\Omega$ and $\Delta$ are simply connected, then $a(\Omega)$, $a(\Delta)\in[6,8]$, so that $3/4\le a(\Delta)/a(\Omega)\le 4/3$. Are these the best constants in general? Can one obtain a constant upper bound for $|a(\Delta)-a(\Omega)|$?

## References


[BG] A.F. Beardon and F.W. Gehring, *Schwarzian derivatives, the Poincaré metric and the kernel function*, Commentarii Mathematici Helvetici **55** (1980), 50–64.

[BS] P.R. Beesack and B. Schwarz, *On the zeros of solutions of second-order linear differential equations*, Canadian Journal of Mathematics **8** (1956), 504–515.

[Ha] R. Harmelin, *Locally convex functions and the Schwarzian derivative*, Israel Journal of Mathematics **67** (1989), 367–379.

[HM] R. Harmelin and D. Minda, *Quasi-invariant domain constants*, Israel Journal of Mathematics **77** (1992), 115–127.

[Hi] J.R. Hilditch, *The hyperbolic metric and the distance to the boundary in plane domains*, unpublished manuscript, *circa* 1985.

[J] V. Jørgensen, *On an inequality for the hyperbolic measure and its applications to the theory of functions*, Mathematica Scandinavica **4** (1956), 113–124.

[K] I. Kra, *Automorphic Forms and Kleinian Groups*, Benjamin, Reading, Mass., 1972.

[MM$_1$] W. Ma and D. Minda, *Euclidean linear invariance and uniform local convexity*, Journal of the Australian Mathematical Society (Series A) **52** (1992), 401–418.

[MM$_2$] W. Ma and D. Minda, *Hyperbolically convex functions*, Annales Polonici Mathematici, to appear.

[$M_1$] D. Minda, *The Schwarzian derivative and univalence criteria*, Contemporary Mathematics **38** (1985), 43–52.

[$M_2$] D. Minda, *Applications of hyperbolic convexity to euclidean and spherical convexity*, Journal d'Analyse Mathématique **49** (1987), 90–105.

[N] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bulletin of the American Mathematical Society **55** (1949), 545–551.

[O] B. Osgood, *Some properties of $f''/f'$ and the Poincaré metric*, Indiana University Mathematics Journal **31** (1982), 449–461.

[$P_1$] Ch. Pommerenke, *Linear-invariante Familien analytischer Funktionen I*, Mathematische Annalen **155** (1964), 108–154.

[$P_2$] Ch. Pommerenke, *Uniformly perfect sets and the Poincaré metric*, Archiv der Mathematik **32** (1979), 192–199.

[S] B. Schwarz, *Complex non-oscillation theorems and criteria of univalence*, Transactions of the American Mathematical Society **80** (1955), 159–186.

[$Y_1$] S. Yamashita, *Univalent analytic functions and the Poincaré metric*, Kodai Mathematical Journal **13** (1990), 164–175.

[$Y_2$] S. Yamashita, *La dérivée d'une fonction univalente dans un domaine hyperbolique*, Comptes Rendus de l'Académie des Sciences, Paris **314** (1992), 45–48.